

Blow-up behavior for the Klein-Gordon and other perturbed semilinear wave equations

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Abstract

We give blow-up results for the Klein-Gordon equation and other perturbations of the semilinear wave equations with superlinear power nonlinearity, in one space dimension or in higher dimension under radial symmetry outside the origin.

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1 Introduction

We consider one dimensional solutions and higher dimensional radial solutions of the following Klein-Gordon equation:

$$\begin{cases} \partial_t^2 U = \Delta U + |U|^{p-1}U - U, \\ U(0) = U_0 \text{ and } U_t(0) = U_1, \end{cases} \quad (1.1)$$

where $U(t) : x \in \mathbb{R}^N \rightarrow U(x, t) \in \mathbb{R}$, $U_0 \in H_{\text{loc},u}^1$ and $U_1 \in L_{\text{loc},u}^2$. The space $L_{\text{loc},u}^2$ is the set of all v in L_{loc}^2 such that

$$\|v\|_{L_{\text{loc},u}^2} \equiv \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2} < +\infty,$$

and the space $H_{\text{loc},u}^1 = \{v \mid v, \nabla v \in L_{\text{loc},u}^2\}$.

The nonlinear Klein Gordon equation appears as a model of self-focusing waves in nonlinear optics (see Bizoń, Chamj and Szpak [4]).

More generally, we consider the following semilinear wave equation:

$$\begin{cases} \partial_t^2 U = \Delta U + |U|^{p-1}U + f(U) + g(|x|, t, \nabla U \cdot \frac{x}{|x|}, \partial_t U), \\ U(0) = U_0 \text{ and } U_t(0) = U_1. \end{cases} \quad (1.2)$$

We assume that the functions f and g are \mathcal{C}^1 functions, where $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfy the following conditions

$$\begin{aligned} (H_f) \quad & |f(u)| \leq M(1 + |u|^q), \quad \text{for all } y \in \mathbb{R} \quad \text{with } (q < p, \ M > 0), \\ (H_g) \quad & |g(X, t, v, z)| \leq M(1 + |v| + |z|), \quad \text{for all } X, t, v, z \in \mathbb{R} \quad \text{with } (M > 0). \end{aligned}$$

We assume also that the function g is globally Lipschitz. Finally, we assume that

$$1 < p \text{ and } p \leq 1 + \frac{4}{N-1} \text{ if } N \geq 2. \quad (1.3)$$

Since U is radial if $N \geq 2$, we introduce

$$u(r, t) = U(r, t) \text{ for } r \in \mathbb{R}, \text{ if } N = 1 \quad (1.4)$$

and

$$u(r, t) = U(x, t) \text{ if } r = |x| \text{ and } N \geq 2 \quad (1.5)$$

and rewrite (1.2) as

$$\begin{cases} \partial_t^2 u = \partial_r^2 u + \frac{(N-1)}{r} \partial_r u + |u|^{p-1}u + f(u) + g(r, t, \partial_r u, \partial_t u), \\ \partial_r u(0, t) = 0 \text{ if } N \geq 2, \\ u(r, 0) = u_0(r) \text{ and } u_t(r, 0) = u_1(r), \end{cases} \quad (1.6)$$

where $u(t) : r \in I \rightarrow u(r, t) \in \mathbb{R}$, with $I = \mathbb{R}^+$ if $N \geq 2$ and $I = \mathbb{R}$ if $N = 1$.

The Cauchy problem of equation (1.2) is solved in $H_{loc,u}^1 \times L_{loc,u}^2$. This follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$ (see for example Georgiev and Todorova [8]), valid whenever $1 < p < 1 + \frac{4}{N-2}$. The existence of blow-up solutions $u(t)$ of (1.2) follows from energy techniques (see for example Levine and Todorova [15]) and Todorova [27]).

If u is a blow-up solution of (1.6), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(r, T(r))\}$ where $r \in I$ such that the maximal influence domain D of u (or the domain of definition of u) is written as

$$D = \{(r, t) \mid r \in I, \ t < T(r)\}. \quad (1.7)$$

Γ is called the blow-up graph of u . A point $r_0 \geq 0$ is a non-characteristic point if there are

$$\delta_0 \in (0, 1) \text{ and } t_0 < T(r_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{r_0, T(r_0), \delta_0} \cap \{t \geq t_0\} \cap \{r \in I\} \quad (1.8)$$

where $\mathcal{C}_{\bar{r}, \bar{t}, \bar{\delta}} = \{(r, t) \mid t < \bar{t} - \bar{\delta}|r - \bar{r}|\}$. We denote by $\mathcal{R} \subset I$ (resp. $\mathcal{S} \subset I$) the set of non-characteristic (resp. characteristic) points.

In the case $(f, g) \equiv (0, 0)$, equation (1.2) reduces to the semilinear wave equation:

$$\begin{cases} \partial_t^2 U = \Delta U + |U|^{p-1}U, \\ U(0) = U_0 \text{ and } U_t(0) = U_1. \end{cases} \quad (1.9)$$

In a series of papers [20], [21], [24] and [22] (see also the note [23]), Merle and Zaag give a full picture of the blow-up for solutions of (1.9) in one space dimension. Recently, in [7], Côte and Zaag refine some of those results and construct a blow-up solution with a characteristic point a , such that the asymptotic behavior of the solution near $(a, T(a))$ shows a decoupled sum of k solitons with alternate signs. Moreover, in [25], Merle and Zaag extend all their results to higher dimensions in the radial case, outside the origin. Our aim in this work is to generalize the result obtained for equation (1.9) in [25] to equation (1.2). Let us note that all our results and proofs hold in both cases of (1.6) ($N = 1$ and $N \geq 2$). However, the situation is a bit more delicate when $N \geq 2$, since we have to avoid the origin which brings a singular term $\frac{N-1}{r}\partial_r u$ to (1.6). Thus, for completeness, we focus on the case $N \geq 2$ avoiding $r = 0$, and stress the fact that all our results hold in the case $N = 1$, even when $r = 0$, and with no symmetry assumptions.

Throughout this paper, we consider $U(x, t)$ a radial blow-up solution of equation (1.2), and use the notation $u(r, t)$ introduced in (1.4). We proceed in 3 sections:

- in Section 2, we give a new Lyapunov functional for equation (1.6) and bound the solution in the energy space.
- in Section 3, we study \mathcal{R} , in particular the blow-up behavior of the solution and the regularity of the blow-up set there.
- in Section 4, we focus on \mathcal{S} , from the point of view of the blow-up behavior, the regularity of the blow-up set and the construction of a multi-soliton solution.

We are aware that our analysis is a generalization of the radial case of equation (1.9) treated by Merle and Zaag in [25]. For that reason, we will give the statements of the results for equation (1.2) and focus only on how to deal with the new perturbation terms appearing in (1.2). Let us add that, we believe that our contribution is non trivial and introduces a new approach for perturbed problems. Moreover, it proves a bunch of results, especially for the Klein-Gordon equation (1.1).

2 A Lyapunov functional for equation (2.2) and a blow-up criterion in the radial case

We showed in [11] and [10] that the argument of Antonini, Merle and Zaag in [3], [17], [19] and [18] extends through a perturbation method to equation (1.2) with no gradient terms even for non radial solutions. The key idea is to modify the Lyapunov functional of [3] with exponentially small terms and define a new functional which is decreasing in time and gives a blow-up criterion. In [25], Merle and Zaag successfully used our ideas to derive a Lyapunov functional for the radial case with no perturbations (i.e. for equation (1.6) with $(f, g) \equiv (0, 0)$). Here, we further refine our argument in [11] and [10] to derive a Lyapunov functional for equations bearing the two features: the presence of perturbation terms and the radial symmetry. For the reader's convenience, we briefly recall the argument in the following.

Given $r_0 > 0$, we recall the following similarity variables' transformation

$$w_{r_0}(y, s) = (T(r_0) - t)^{\frac{2}{p-1}} u(r, t), \quad y = \frac{r - r_0}{T(r_0) - t}, \quad s = -\log(T(r_0) - t). \quad (2.1)$$

The function $w = w_{r_0}$ satisfies the following equation for all $y \in (-1, 1)$ and $s \geq \max(-\log T(r_0), -\log r_0)$:

$$\begin{aligned} \partial_s^2 w &= \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{y,s}^2 w \\ &+ e^{-s} \frac{(N-1)}{r_0 + ye^{-s}} \partial_y w + e^{-\frac{2ps}{p-1}} f\left(e^{\frac{2s}{p-1}} w\right) \\ &+ e^{-\frac{2ps}{p-1}} g\left(r_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{p-1}} \partial_y w, e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \partial_y w + \frac{2}{p-1} w)\right), \end{aligned} \quad (2.2)$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y (\rho(1-y^2) \partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \quad (2.3)$$

In the whole paper, we denote

$$F(u) = \int_0^u f(v) dv. \quad (2.4)$$

Let us recall that for the case $(f, g) \equiv (0, 0)$, the Lyapunov functional in one space dimension is

$$E_0(w(s)) = \int_{-1}^1 \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy, \quad (2.5)$$

which is defined in the Hilbert space

$$\mathcal{H} = \left\{ q \in H_{\text{loc}}^1 \times L_{\text{loc}}^2(-1, 1) \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 \left(q_1^2 + (q_1')^2 (1 - y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (2.6)$$

Introducing

$$E(w(s), s) = E_0(w(s)) + I(w(s), s) + J(w(s), s) \quad (2.7)$$

where,

$$I(w(s), s) = -e^{-\frac{2(p+1)s}{p-1}} \int_{-1}^1 F(e^{\frac{2s}{p-1}} w) \rho dy, \quad (2.8)$$

$$J(w(s), s) = -e^{-\gamma s} \int_{-1}^1 w \partial_s w \rho dy, \quad (2.9)$$

with

$$\gamma = \min\left(\frac{1}{2}, \frac{p-q}{p-1}\right) > 0, \quad (2.10)$$

we claim the following:

Proposition 2.1 (A new functional for equation (2.2))

(i) *There exists $C = C(p, N, M, q) > 0$ and $S_0(p, N, M, q) \in \mathbb{R}$ such that for all $r_0 > 0$ and for all $s \geq \max(-\log T(r_0), S_0, -4 \log r_0, -\log \frac{r_0}{2})$,*

$$\frac{d}{ds} E(w_{r_0}(s), s) \leq \frac{p+3}{2} e^{-\gamma s} E(w_{r_0}(s), s) - \frac{3}{p-1} \int_{-1}^1 (\partial_s w_{r_0})^2 \frac{\rho}{1-y^2} dy + C e^{-2\gamma s}. \quad (2.11)$$

(ii) **(A blow-up criterion)** *There exists $S_1(p, N, M, q) \in \mathbb{R}$ such that, for all $s \geq \max(s_0, S_1)$, we have $H(w(s), s) \geq 0$.*

Remark: From (i), we see that the Lyapunov functional for equation (2.2) is in fact $H(w_{r_0}(s), s)$ where

$$H(w(s), s) = E(w(s), s) e^{\frac{p+3}{2\gamma} e^{-\gamma s}} + \mu e^{-2\gamma s} \quad (2.12)$$

not $E(w_{r_0}(s), s)$ nor $E_0(w_{r_0}(s))$, for some large constant μ .

Remark: We already know from [11] and [10] that even in the non-radial setting, equation (1.6) has a Lyapunov functional given by a perturbed form of a natural extension to higher dimensions of $E(w_{r_0}(s), s)$ (2.5). Unfortunately, already for the non-perturbed case of (1.2) with $(f, g) \equiv (0, 0)$, due to the lack of information on stationary solutions in similarity variables in dimensions $N \geq 2$, it wasn't possible to go further in the analysis, and the investigation had to stop at the step of bounding

the solution in similarity variables. On the contrary, when $N = 1$, Merle and Zaag could obtain a very precise characterization of blow-up in [20], [21], [24], [22] (with some refinements by Côte and Zaag in [7]).

Here, considering perturbations as stated in (1.2) and restricting ourselves to one dimensional solutions or higher dimensional radial solutions, we find a different Lyapunov functional. Considering arbitrary blow-up points in one-space dimension (including the origin) and any non-zero blow-up in higher dimensions, the characterization of stationary solutions in one space dimension is enough, and we are able to go in our analysis as far as in the one-dimensional case.

Following [17] and [20], together with our techniques to handle perturbations in [11] and [10], we derive with no difficulty the following:

Proposition 2.2 (Boundedness of the solutions of equation (2.2) in the energy space) *For all $r_0 > 0$, there is a $C_2(r_0) > 0$ and $S_2(r_0) \in \mathbb{R}$ such that for all $r \in [\frac{r_0}{2}, \frac{3r_0}{2}]$ and $s \geq S_2(r_0)$,*

$$\int_{-1}^1 ((\partial_y w_r)^2 (1 - y^2) + (w_r)^2 + (\partial_s w_r)^2 + |w_r|^{p+1}) \rho dy \leq C_2(r_0).$$

Proof: The adaptation is straightforward from [17] and Proposition 3.5 page 66 in [20]. The only difference is in the justification of the limit at infinity of $E_0(w_{r_0}(s))$, which follows from the limit of $H(w_{r_0}(s), s)$ defined in (2.12). In fact, we know from Proposition 2.1 that $H(w_{r_0}(s), s)$ is decreasing and bounded from below, and such an information is unavailable for $E_0(w_{r_0}(s))$. ■

Proof of Proposition 2.1:

(i) Consider $r_0 > 0$, $s \geq \max(-\log T(r_0), 0, -\log \frac{r_0}{2}, -4 \log r_0)$ and write $w = w_{r_0}$ for simplicity. From the similarity variables' transformation (2.1), we see that

$$r = r_0 + ye^{-s} \in \left[\frac{r_0}{2}, \frac{3r_0}{2} \right]. \quad (2.13)$$

Multiplying equation (2.2) by $\partial_s w \rho$ and integrating for $y \in (-1, 1)$, we see by (2.5) and (2.8) that

$$\begin{aligned} \frac{d}{ds}(E_0(w(s)) + I(w(s), s)) &= \frac{-4}{p-1} \int_{-1}^1 \frac{(\partial_s w)^2}{1-y^2} \rho dy + \underbrace{(N-1)e^{-s} \int_{-1}^1 \partial_s w \partial_y w \frac{\rho}{r} dy}_{I_1(s)} \\ &\quad + \underbrace{\frac{2(p+1)}{p-1} e^{-\frac{2(p+1)s}{p-1}} \int_{-1}^1 F\left(e^{\frac{2s}{p-1}} w\right) \rho dy}_{I_2(s)} + \underbrace{\frac{2}{p-1} e^{-\frac{2ps}{p-1}} \int_{-1}^1 f\left(e^{\frac{2s}{p-1}} w\right) w \rho dy}_{I_3(s)} \\ &\quad + \underbrace{e^{-\frac{2ps}{p-1}} \int_{-1}^1 g\left(r_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{p-1}} \partial_y w, e^{\frac{(p+1)s}{p-1}} \left(\partial_s w + y \partial_y w + \frac{2w}{p-1}\right)\right) \partial_s w \rho dy}_{I_4(s)}. \end{aligned}$$

where r is defined in (2.13). Using (2.13), we write

$$|I_1(s)| \leq Ce^{-s} \int_{-1}^1 (\partial_y w)^2 \rho (1-y^2) dy + \frac{Ce^{-s}}{r_0^2} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy. \quad (2.14)$$

Using the fact that

$$|F(x)| + |xf(x)| \leq C(1 + |x|^{q+1}) \leq C(1 + |x|^{p+1}), \quad (2.15)$$

where F and f are defined in (2.4) and (1.2), we obtain that

$$|I_2(s)| + |I_3(s)| \leq Ce^{-\frac{2(p-q)s}{p-1}} + Ce^{-\frac{2(p-q)s}{p-1}} \int_{-1}^1 |w|^{p+1} \rho dy. \quad (2.16)$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ and the hypothesis (H_g) , we write that

$$|I_4(s)| \leq Ce^{-s} \int_{-1}^1 \left((\partial_s w)^2 + (w^2) \right) \rho dy + Ce^{-s} \int_{-1}^1 |\partial_y w| |\partial_s w| \rho dy + Ce^{-s}. \quad (2.17)$$

Similarly, we prove that

$$\int_{-1}^1 |\partial_y w| |\partial_s w| \rho dy \leq \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + \int_{-1}^1 (\partial_y w)^2 (1-y^2) \rho dy. \quad (2.18)$$

Combining (2.17) and (2.18), we conclude that

$$|I_4(s)| \leq Ce^{-s} \int_{-1}^1 \left((\partial_y w)^2 (1-|y|^2) + \frac{(\partial_s w)^2}{1-y^2} + w^2 \right) \rho dy + Ce^{-s}. \quad (2.19)$$

Then, by using (??), (2.14), (2.16) and (2.19), we deduce that

$$\begin{aligned} \frac{d}{ds} (E_0(w(s)) + I(w(s), s)) &\leq \left(-\frac{4}{p-1} + Ce^{-\frac{s}{2}} \right) \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy \\ &\quad + Ce^{-s} \int_{-1}^1 \left((\partial_y w)^2 (1-|y|^2) + w^2 \right) \rho dy \\ &\quad + Ce^{-2\gamma s} \int_{-1}^1 |w|^{p+1} \rho dy + Ce^{-2\gamma s}. \end{aligned} \quad (2.20)$$

Considering $J(w(s), s)$ defined in (2.9), we obtain from equation (2.2) and integration by parts

$$\begin{aligned} e^{\gamma s} \frac{d}{ds} J(w(s), s) &= - \int_{-1}^1 (\partial_s w)^2 \rho dy + \int_{-1}^1 (\partial_y w)^2 (1-y^2) \rho dy + \frac{2p+2}{(p-1)^2} \int_{-1}^1 w^2 \rho dy \\ &\quad - \int_{-1}^1 |w|^{p+1} \rho dy + \left(\gamma + \frac{p+3}{p-1} - 2N \right) \int_{-1}^1 w \partial_s w(s) \rho dy - 2 \int_{-1}^1 w \partial_s w y \rho' dy \\ &\quad - 2 \int_{-1}^1 \partial_s w \partial_y w y \rho dy - e^{-\frac{2ps}{p-1}} \int_{-1}^1 w f \left(e^{\frac{2s}{p-1}} w \right) \rho dy - (N-1) e^{-s} \int_{-1}^1 w \partial_y w \frac{\rho}{r} dy \\ &\quad - e^{-\frac{2ps}{p-1}} \int_{-1}^1 w g \left(r_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{p-1}} \partial_y w, e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \partial_y w + \frac{2}{p-1} w) \right) \rho dy. \end{aligned}$$

Combining (2.7), (2.8) and (??), we write

$$\begin{aligned}
e^{\gamma s} \frac{d}{ds} J(w(s), s) &\leq \frac{p+3}{2} (E_0(w(s)) + I(w(s), s)) - \frac{p-1}{4} \int_{-1}^1 (\partial_y w)^2 (1-y^2) \rho dy \\
&\quad - \frac{p+1}{2(p-1)} \int_{-1}^1 w^2 \rho dy - \frac{p-1}{2(p+1)} \int_{-1}^1 |w|^{p+1} \rho dy \\
&\quad + \underbrace{(\gamma + \frac{p+3}{p-1} - 2N + \frac{p+3}{2} e^{-\gamma s}) \int_{-1}^1 w \partial_s w \rho dy}_{J_1(s)} \\
&\quad + \underbrace{\frac{8}{p-1} \int_{-1}^1 w \partial_s w \frac{y^2}{1-y^2} \rho dy}_{J_2(s)} - \underbrace{2 \int_{-1}^1 \partial_s w \partial_y w y \rho dy}_{J_3(s)} - \underbrace{e^{-\frac{2ps}{p-1}} \int_{-1}^1 w f\left(e^{\frac{2s}{p-1}} w\right) \rho dy}_{J_4(s)} \\
&\quad - \underbrace{e^{-\frac{2ps}{p-1}} \int_{-1}^1 w g\left(r_0 + y e^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{p-1}} \partial_y w, e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \partial_y w + \frac{2}{p-1} w)\right) \rho dy}_{J_5(s)} \\
&\quad + \underbrace{\frac{p+3}{2} e^{-\frac{2(p+1)s}{p-1}} \int_{-1}^1 F(e^{\frac{2}{p-1}s} w) \rho dy}_{J_6(s)} - \underbrace{(N-1) e^{-s} \int_{-1}^1 w \partial_y w \frac{\rho}{r} dy}_{J_7(s)}.
\end{aligned} \tag{2.21}$$

We now study each of the last five terms. To estimate $J_1(s)$ and $J_2(s)$, we use the Cauchy-Schwartz inequality to have

$$|J_1(s)| \leq C e^{\frac{\gamma s}{2}} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + C e^{-\frac{\gamma s}{2}} \int_{-1}^1 w^2 \rho dy. \tag{2.22}$$

$$|J_2(s)| \leq C e^{\frac{\gamma s}{2}} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + C e^{-\frac{\gamma s}{2}} \int_{-1}^1 w^2 \frac{y^2 \rho}{1-y^2} dy.$$

Recalling the following Hardy-Sobolev estimate (see Appendix B page 1163 in [17] for the proof):

$$\int_{-1}^1 h^2 \frac{\rho}{1-y^2} dy \leq C \int_{-1}^1 h^2 \rho dy + C \int_{-1}^1 (h'(y))^2 \rho (1-y^2) dy, \tag{2.23}$$

we conclude that

$$\begin{aligned}
|J_2(s)| &\leq C e^{\frac{\gamma s}{2}} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + C e^{-\frac{\gamma s}{2}} \int_{-1}^1 w^2 \rho dy \\
&\quad + C e^{-\frac{\gamma s}{2}} \int_{-1}^1 (\partial_y w)^2 \rho (1-y^2) dy.
\end{aligned} \tag{2.24}$$

Using the Cauchy-Schwartz inequality, we have

$$|J_3(s)| \leq Ce^{\frac{\gamma s}{2}} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + Ce^{-\frac{\gamma s}{2}} \int_{-1}^1 (\partial_y w)^2 \rho(1-y^2) dy. \quad (2.25)$$

From (2.15), we write

$$|J_4(s)| + |J_6(s)| \leq Ce^{-\gamma s} + Ce^{-\gamma s} \int_{-1}^1 |w|^{p+1} \rho dy. \quad (2.26)$$

In a similar way, using the hypothesis (H_g) and (2.23), we have

$$\begin{aligned} |J_5(s)| &\leq Ce^{-\gamma s} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + Ce^{-\gamma s} \int_{-1}^1 (\partial_y w)^2 \rho(1-y^2) dy \\ &\quad + Ce^{-\gamma s} \int_{-1}^1 w^2 \rho dy + Ce^{-\gamma s}. \end{aligned} \quad (2.27)$$

Using (2.13) and (2.23), we write

$$|J_7(s)| \leq Ce^{-\frac{s}{2}} \int_{-1}^1 (\partial_y w)^2 \rho(1-y^2) dy + Ce^{-\frac{s}{2}} \int_{-1}^1 w^2 \rho dy. \quad (2.28)$$

Finally, using (2.21), (2.22), (2.24), (2.25), (2.27) and (2.28), we deduce that

$$\begin{aligned} e^{\gamma s} \frac{d}{ds} J(w(s), s) &\leq \frac{p+3}{2} \left(E_0(w(s)) + I(w(s), s) \right) \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p-1}{4} \right) \int_{-1}^1 (\partial_y w)^2 (1-y^2) \rho dy \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p+1}{2(p-1)} \right) \int_{-1}^1 w^2 \rho dy \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p-1}{2(p+1)} \right) \int_{-1}^1 |w|^{p+1} \rho dy \\ &\quad + Ce^{\frac{\gamma s}{2}} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy + Ce^{-\gamma s}. \end{aligned} \quad (2.29)$$

From (2.20) and (2.29), we obtain

$$\begin{aligned} \frac{d}{ds} E(w(s), s) &\leq Ce^{-2\gamma s} + \frac{p+3}{2} e^{-\gamma s} E(w(s), s) \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{4}{p-1} \right) \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p+1}{2(p-1)} \right) e^{-\gamma s} \int_{-1}^1 w^2 \rho dy \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p-1}{4} \right) e^{-\gamma s} \int_{-1}^1 (\partial_y w)^2 (1-|y|^2) \rho dy \\ &\quad + \left(Ce^{-\frac{\gamma s}{2}} - \frac{p-1}{2(p+1)} \right) e^{-\gamma s} \int_{-1}^1 |w|^{p+1} \rho dy. \end{aligned} \quad (2.30)$$

We now choose $S_0 \geq 0$, large enough, so that for all $s \geq S_0$, we have

$$\frac{p-1}{4} - Ce^{-\frac{\gamma s}{2}} \geq 0, \quad \frac{p+1}{2(p-1)} - Ce^{-\frac{\gamma s}{2}} \geq 0, \quad \frac{p-1}{2(p+1)} - Ce^{-\frac{\gamma s}{2}} \geq 0, \quad \frac{1}{p-1} - Ce^{-\frac{\gamma s}{2}} \geq 0.$$

Then, we deduce that, for all $s \geq \max(S_0, -\log T(r_0), -\log \frac{r_0}{2}, -4 \log r_0)$, we have

$$\frac{d}{ds} E(w(s), s) \leq Ce^{-2\gamma s} + \frac{p+3}{2} e^{-\gamma s} E(w(s), s) - \frac{3}{p-1} \int_{-1}^1 (\partial_s w)^2 \frac{\rho}{1-y^2} dy. \quad (2.31)$$

This yields (i) of Proposition 2.1.

(ii) We finish the proof of Proposition 2.1 here. More precisely, we prove that there exists $S_1(p, N, M, q) \in \mathbb{R}$ such that, for all $x_0 \in \mathbb{R}^N$ and $T_0 \in (0, T(x_0)]$,

$$\forall s \geq \max(-\log T_0, S_1,) \quad H(w_{x_0, T_0}(s), s) \geq 0. \quad (2.32)$$

We give the proof only in the case where x_0 is a non characteristic point. Note that the case where x_0 is a characteristic point can be done exactly as in Appendix A page 119 in [20]. If x_0 is a non characteristic point, the argument is the same as in the corresponding part in [3]. We write the proof for completeness. Arguing by contradiction, we assume that there exists a non characteristic point $x_0 \in \mathbb{R}^N$, $T_0 \in (0, T(x_0)]$ and $s_1 \geq \max(-\log T_0, S_1,)$ such that $H(w(s_1), s_1) < 0$, where $w = w_{x_0, T_0}$. By definition (2.12) of H , we write

$$\begin{aligned} H(W(s), s) &\geq \mu e^{-2\gamma s} - e^{\frac{p+3}{2}e^{-\gamma s}} \left(\frac{1}{p+1} + Ce^{-2\gamma s} \right) \int_{-1}^1 |W|^{p+1} \rho dy \\ &\quad + e^{\frac{p+3}{2}e^{-\gamma s}} \left(\left(\frac{1}{2} - Ce^{-\gamma s} \right) \int_{-1}^1 (\partial_s W)^2 \rho dy + \left(\frac{p+1}{(p-1)^2} - Ce^{-\gamma s} \right) \int_{-1}^1 W^2 \rho dy \right) \\ &\geq -\frac{2}{p+1} \int_{-1}^1 |W|^{p+1} \rho dy. \end{aligned}$$

if $s \geq S_2(p, N, q, M) \geq S_1(p, N, q, M)$ for some $S_2(p, N, q, M) \in \mathbb{R}$ large enough. Using this inequality together with the fact that $H(W(s), s)$ is decreasing by the remark following Proposition 2.1, we see that the argument used by Antonini and Merle in Theorem 2 page 1147 in [3] for the equation (1.9) works here and we get the blow-up criterion. This concludes the proof of Proposition 2.1. \blacksquare

3 Blow-up results related to non-characteristic points

Let us first introduce for all $|d| < 1$ the following solitons defined by

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \quad (3.1)$$

Note that $\kappa(d)$ is a stationary solution of (2.2), in the particular case where $(f, g) \equiv (0, 0)$ and in one space dimension.

Adapting the analysis of [20] and [21], we claim the following:

Theorem 1 (Blow-up behavior and regularity of the blow-up set on \mathcal{R})

(i) **(Regularity related to \mathcal{R})** $\mathcal{R} \neq \emptyset$, $\mathcal{R} \cap \mathbb{R}_+^*$ is an open set, and $x \mapsto T(x)$ is of class C^1 on $\mathcal{R} \cap \mathbb{R}_+^*$.

(ii) **(Blow-up behavior in similarity variables)** There exist $\mu_0 > 0$ and $C_0 > 0$ such that for all $r_0 \in \mathcal{R} \cap \mathbb{R}_+^*$, there exist $\theta(r_0) = \pm 1$ and $s_0(r_0) \geq -\log T(r_0)$ such that for all $s \geq s_0$:

$$\left\| \begin{pmatrix} w_{r_0}(s) \\ \partial_s w_{r_0}(s) \end{pmatrix} - \theta(r_0) \begin{pmatrix} \kappa(T'(r_0)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)}. \quad (3.2)$$

Moreover, $E_0(w_{r_0}(s)) \rightarrow E_0(\kappa_0)$ as $s \rightarrow \infty$.

Remark: As stated in the introduction, this result holds also when $N = 1$, with no symmetry assumptions an initial data, for all $r_0 \in \mathbb{R}$, even $r_0 = 0$; when $N \geq 2$ and if $0 \in \mathcal{R}$, the asymptotic behavior of w_0 remains open.

Proof: As in the non-perturbed radial case (take $(f, g) \equiv (0, 0)$ in (1.6)) treated in [25], we need to make some minor adaptations to the one-dimensional non-perturbed case treated in [20] and [21]. It happens that the same adaptation pattern works in the present case, and that is the reason why we don't mention it, and refer the reader to the proof of Theorem 1 in page 358 of [25]. The only points to check are the following:

- *Continuity with respect to the scaling parameter:* Due to the fact that equation (1.6) is no longer invariant under the scaling

$$\lambda \mapsto u_\lambda(\xi, \tau) = \lambda^{\frac{2}{p-1}} u(\lambda\xi, \lambda\tau),$$

we need to understand the continuous dependence of the solutions of the following family of equations

$$\begin{aligned} \partial_t^2 u = \partial_r^2 u + \frac{(N-1)\lambda}{x + \lambda r} \partial_r u + |u|^{p-1} u + \lambda^{\frac{2p}{p-1}} f\left(\lambda^{\frac{-2}{p-1}} u\right) \\ + \lambda^{\frac{2p}{p-1}} g\left(\lambda r, \lambda t, \lambda^{-\frac{p+1}{p-1}} \partial_r u, \lambda^{-\frac{p+1}{p-1}} \partial_t u\right), \end{aligned} \quad (3.3)$$

with respect to initial data and the parameters $x \geq 0$ and $\lambda > 0$ (including the limit as $\lambda \rightarrow 0$ and this is a classical estimate.

- *A new statement for the trapping result:* This is due to the fact that equation (2.2) in similarity variables depends on a parameter $r_0 > 0$ and contains new terms of order $e^{-\gamma s}$ (γ is defined in (2.10)) (it is no longer autonomous). This is the trapping result in our setting:

Theorem 2 (Trapping near the set of non zero stationary solutions of (2.2))

For all $\rho_0 > 0$, there exist positive ϵ_0 , μ_0 and C_0 such that for all $\epsilon^* \leq \epsilon_0$, there exists $s_0(\epsilon^*)$ such that if $r_0 \geq \rho_0$, $s^* \geq s_0$ and $w \in C([s^*, \infty), \mathcal{H})$ is a solution of equation (2.2) with

$$\forall s \geq s^*, \quad E(w(s), s) \geq E_0(\kappa_0) - e^{-\frac{\gamma s}{2}}, \quad (3.4)$$

and

$$\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^*$$

for some $d^* \in (-1, 1)$ and $\omega^* = \pm 1$, then there exists $d_\infty \in (-1, 1)$ such that

$$|\arg \tanh d_\infty - \arg \tanh d^*| \leq C_0 \epsilon^*,$$

and for all $s \geq s^*$,

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \epsilon^* e^{-\mu_0(s-s^*)}.$$

Proof. The proof follows the pattern of the radial case treated in [25]. For that reason, we refer the reader to the Proof of Theorem 2 page 360 in that paper, and focus in the following only on how to treat the new terms generated by the perturbations f and g in (1.2). With respect to the pure power case in one space dimension, the difference comes from the linearization of (2.2) around the stationary solutions $\kappa(d, y)$ in (3.1), where we see the following lower order terms:

$$\begin{aligned} & \left| \frac{(N-1)e^{-s}}{r_0 + ye^{-s}} \partial_y w \right| \leq \frac{2}{\rho_0} (N-1)e^{-s} |\partial_y w|; \\ & e^{-\frac{2ps}{p-1}} \left| f \left(e^{\frac{2s}{p-1}} w \right) \right| \leq CM e^{-\frac{2(p-q)s}{p-1}} + CM e^{-\frac{2(p-q)s}{p-1}} |w|^p; \\ & e^{-\frac{2ps}{p-1}} \left| g \left(r_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{p-1}} \partial_y w, e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \partial_y w + \frac{2}{p-1} w) \right) \right| \\ & \leq CM e^{-s} \left(1 + |\partial_s w| + |\partial_y w| + |w| \right). \end{aligned} \quad (3.5)$$

as soon as $r_0 \geq \rho_0 > 0$, and $s \geq -\log \frac{\rho_0}{2}$. For more details an the adaptation, we refer the reader to the proof of Theorem 1 in page 358 of [25]. \blacksquare

4 Blow-up results related to characteristic points

The first question in this case is of course the existence of examples of initial data with $\mathcal{S} \neq \emptyset$. If the perturbation g introduced in (1.2) does not depend on $|x|$, then the existence of such an example follows from the knowledge of the blow-up behavior at non-characteristic points, as in the pure power nonlinearity case (1.9). If g

depends on $|x|$, then we need to apply the constructive method of Côte and Zaag [7], which relies fundamentally on the knowledge of the blow-up behavior near a characteristic point. For that reason, we leave the existence issues to the end of the section, and start with the description of the blow-up features near characteristic points. More precisely, we proceed in two sections:

- In Section 4.1, we consider arbitrary blow-up solutions having a non-zero characteristic point, and we give a full description of its blow-up behavior and its blow-up set near this characteristic point.
- In Section 4.2, we prove the existence of such a solution, and also give some criteria for the existence or the non-existence of characteristic points.

4.1 Description of the blow-up behavior and the blow-up set near a characteristic point

Now, given $r_0 \in \mathcal{S} \cap \mathbb{R}_+^*$, we have the same description for the asymptotic of w_{r_0} as in the one-dimensional case with no perturbations (i.e. for equation (1.6) with $(f, g) \equiv (0, 0)$) refined recently by Côte and Zaag in [7]. In order to state the result, let us introduce

$$\bar{\zeta}_i(s) = \left(i - \frac{(k+1)}{2} \right) \frac{(p-1)}{2} \log s + \bar{\alpha}_i(p, k) \quad (4.1)$$

where the sequence $(\bar{\alpha}_i)_{i=1, \dots, k}$ is uniquely determined by the fact that $(\bar{\zeta}_i(s))_{i=1, \dots, k}$ is an explicit solution with zero center of mass for the following ODE system:

$$\frac{1}{c_1} \dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)}, \quad (4.2)$$

where $c_1 = c_1(p) > 0$ and $\zeta_0(s) \equiv \zeta_{k+1}(s) \equiv 0$ (see Section 2 in [7] for a proof of this fact). Note that $c_1 = c_1(p) > 0$ is a constant appearing in system (4.11), itself inherited from Proposition 3.2 of [24]. With this definition, we can state our result (for the statement in one space dimension, see Theorem 6 in [24] and Theorem 1 in [7]):

Theorem 3 (Description of the behavior of w_{r_0} where r_0 is characteristic)
Consider $r_0 \in \mathcal{S} \cap \mathbb{R}_+^$. Then, there is $\zeta_0(r_0) \in \mathbb{R}$ such that*

$$\left\| \begin{pmatrix} w_{r_0}(s) \\ \partial_s w_{r_0}(s) \end{pmatrix} - \theta_1 \begin{pmatrix} \sum_{i=1}^{k(r_0)} (-1)^{i+1} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0 \text{ and } E_0(w_{r_0}(s)) \rightarrow k(r_0) E_0(\kappa_0) \quad (4.3)$$

as $s \rightarrow \infty$, for some

$$k(r_0) \geq 2, \quad (4.4)$$

$$\theta_i = \theta_1 (-1)^{i+1}, \quad \theta_1 = \pm 1 \quad (4.5)$$

and continuous $d_i(s) = -\tanh \zeta_i(s)$ with

$$\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0, \quad (4.6)$$

where $\bar{\zeta}_i(s)$ is introduced above in (4.1).

Remark: As stated in the introduction, this result holds also when $N = 1$, with no symmetry assumptions an initial data, for all $r_0 \in \mathbb{R}$, even $r_0 = 0$; when $N \geq 2$ and if $0 \in \mathcal{S}$, the asymptotic behavior of w_0 remains open.

Proof: As in the one-dimensional case with no perturbations (i.e. for equation (1.6) with $(f, g) \equiv (0, 0)$), the proof of the asymptotic behavior and the geometric results on \mathcal{S} (see Theorem 5 below) go side by side. Evidently the refined description given by (4.6) is obtained as in [7]. We leave the proof after the statement of Theorem 5. ■

Let us note that we get the following result on the energy behavior from the asymptotic behavior at a non-characteristic point (see (ii) of Theorem 1) and at a characteristic point (see Theorem 3):

Corollary 4 (A criterion for non-characteristic points)

For all $r_0 > 0$, there exist $C_3(r_0) > 0$ and $S_3(r_0) \in \mathbb{R}$ such that:

(i) For all $r \in [\frac{r_0}{2}, \frac{3r_0}{2}]$ and $s \geq S_3$, we have

$$E_0(w_r(s)) \geq k(r)E_0(\kappa_0) - C_3(r_0)e^{-\gamma s}.$$

(ii) If for some $r \in [\frac{r_0}{2}, \frac{3r_0}{2}]$ and $s \geq S_3$, we have

$$E_0(w_r(s)) < 2E_0(\kappa_0) - C_3(r_0)e^{-\gamma s},$$

then $r \in \mathcal{R}$.

Remark: With respect to the statement in one-space dimensions with no perturbations, (Corollary 7 in [24]), this statement has additional exponentially small terms. This comes from the fact that the functional $E(w(s), s)$ is no longer decreasing, and that one has to work instead with the functional $H(w(s), s)$ (2.12) which is decreasing, and differs from $E(w(s), s)$ by exponentially small terms, uniformly controlled for $r \in [\frac{r_0}{2}, \frac{3r_0}{2}]$ thanks to the uniform estimates of Proposition 2.2.

Proof: If one replaces $E(w(s), s)$ by $H(w(s), s)$, then the proof is straightforward from Theorems 1 and 3 together with the monotonicity of $H(w(s), s)$ (see (2.12) and (2.1)). Since the difference between the two functionals is exponentially small, uniformly for $r \in [\frac{r_0}{2}, \frac{3r_0}{2}]$ (see (2.12), (2.11) and Proposition 2.2), we get the conclusion of Corollary 4. ■

Finally, we give in the following some geometric information related to characteristic points (for the statement in one space dimension, see Theorem 1, Theorem 2 and the following remark in [22]):

Theorem 5 (Geometric considerations on \mathcal{S})

- (i) **(Isolatedness of characteristic points)** Any $r_0 \in \mathcal{S} \cap \mathbb{R}_+^*$ is isolated.
(ii) **(Corner shape of the blow-up curve at characteristic points)** If $r_0 \in \mathcal{S} \cap \mathbb{R}_+^*$ with $k(r_0)$ solitons and $\zeta_0(r_0) \in \mathbb{R}$ as center of mass of the solitons' center as shown in (4.3) and (4.6), then

$$T'(r) + \theta(r) \sim \frac{\theta(r)\nu e^{-2\theta(r)\zeta_0(r_0)}}{|\log|r - r_0||^{\frac{(k(r_0)-1)(p-1)}{2}}} \quad (4.7)$$

$$T(r) - T(r_0) + |r - r_0| \sim \frac{\nu e^{-2\theta(r)\zeta_0(r_0)}|r - r_0|}{|\log|r - r_0||^{\frac{(k(r_0)-1)(p-1)}{2}}} \quad (4.8)$$

as $r \rightarrow r_0$, where $\theta(r) = \frac{r-r_0}{|r-r_0|}$ and $\nu = \nu(p) > 0$.

Proof: See below.

Remark: As stated in the remark after Theorem 3, our result holds for $N = 1$, with no symmetry assumptions an initial data, for all $r_0 \in \mathbb{R}$, even $r_0 = 0$; when $N \geq 2$, and if $0 \in \mathcal{S}$, the asymptotic behavior of w_0 remains open.

Proof: As in the one-dimensional case with no perturbations

Remark: Note from (i) that the multi-dimensional version $U(x, t) = u(|x|, t)$ has a finite number of concentric spheres of characteristic points in the set $\{\frac{1}{R} < |x| < R\}$ for every $R > 1$. This is consistent with our conjecture in [22] where we guessed that in dimension $N \geq 2$, the $(N - 1)$ -dimensional Hausdorff measure of \mathcal{S} is bounded in compact sets of \mathbb{R}^N . Note that this conjecture is related to the result of Velázquez who proved in [28] that the $(N - 1)$ -dimensional Hausdorff measure of the blow-up set for the semilinear heat equation with subcritical power nonlinearity is bounded in compact sets of \mathbb{R}^N .

As a consequence of our analysis, particularly the lower bound on $T(r)$ in (4.8), we have the following estimate on the blow-up speed in the backward light cone with vertex $(r_0, T(r_0))$ where $r_0 > 0$ (for the statement in one space dimension, see Corollary 3 in [22]):

Corollary 6 (Blow-up speed in the backward light cone) For all $r_0 > 0$, there exists $C_4(r_0) > 0$ such that for all $t \in [0, T(r_0))$, we have

$$\frac{|\log(T(r_0) - t)|^{\frac{k(r_0)-1}{2}}}{C_4(r_0)(T(r_0) - t)^{\frac{2}{p-1}}} \leq \sup_{|x-r_0| < T(r_0)-t} |u(x, t)| \leq \frac{C_4(r_0)|\log(T(r_0) - t)|^{\frac{k(r_0)-1}{2}}}{(T(r_0) - t)^{\frac{2}{p-1}}}.$$

Remark: Note that when $r_0 \in \mathcal{R} \cap \mathbb{R}_+^*$, the blow-up rate of u in the backward light cone with vertex $(r_0, T(r_0))$ is given by the solution of the associated ODE $u'' = u^p$. When $r_0 \in \mathcal{S} \cap \mathbb{R}_+^*$, the blow-up rate is higher and quantified, according to $k(r_0)$, the number of solitons appearing in the decomposition (4.3).

Proof: When $r_0 \in \mathcal{R}$, the result follows from the fact that the convergence in (3.2)

is true also in $L^\infty \times L^2$ from (3.2) and the Sobolev embedding in one dimension. When $r_0 \in \mathcal{S}$, see the proof of Corollary 3 of [22] given in Section 3.3 of that paper. ■

Proof of Theorems 3 and 5: The proof follows the pattern of the original proof, given in [20], [24], [22] and [7]. In the following, we recall its different parts.

Part 1: Proof of (4.3) without (4.4) nor (4.5) and with the estimate

$$\zeta_{i+1}(s) - \zeta_i(s) \rightarrow \infty \text{ as } s \rightarrow \infty \quad (4.9)$$

instead of (4.6) (note that (4.9) is meaningful only when $k(r_0) \geq 2$).

The original statement of this part is given in Theorem 2 (B) page 47 in [20] and the proof in section 3.2 page 66 in that paper. Note that this part doesn't exclude the possibility of having $k(r_0) = 0$ or $k(r_0) = 1$. The adaptation is straightforward. As in the non-characteristic case above, one has to use the Duhamel formulation in the radial which may be derived from [26].

Part 2: Assuming that (4.4) is true, we prove (4.5) with the estimates

$$\begin{aligned} |\zeta_i(s) - \bar{\zeta}_i(s)| &\leq C, \\ T(r) - T(r_0) + |r - r_0| &\leq \frac{C|r - r_0|}{|\log |r - r_0||^{\frac{(k(r_0)-1)(p-1)}{2}}} \end{aligned} \quad (4.10)$$

instead of (4.6) and (4.8).

The original statement is given in Propositions 3.1 and 3.13 in [24]. The reader has to read Section 3 and Appendices B and C in that paper. The adaptation is straightforward, except for the effect of the new terms in equation (2.2), which produce exponentially small terms in many parts of the proof (see (3.5)). In particular, Lemma 3.11 of [24] has to be changed by adding $Ce^{-\gamma s}$ where γ is defined in (2.10) to the right of all the differential inequalities.

Part 3: Proof of (4.4) and the fact that the interior of \mathcal{S} is empty.

The original statement is given in Proposition 4.1 of [24]. The adaptation is as delicate as in [25]. In particular, it involves the ruling-out of the occurrence of the case where, locally near the origin, the blow-up set of the multi-dimensional version $U(x, t)$ is a forward light cone with vertex $(0, T(0))$ (see Lemma 4.5 page 367 in [25]). As in that paper, the proof of the non-occurrence of this case is based in particular on a local energy estimate by Shatah and Struwe [26]. For the reader's convenience, we adapt in Appendix A that energy estimate to our case (1.2), namely to perturbations of the pure power equation (1.9). For the other arguments, we refer to the corresponding part in [25] (see Part 3 page 336 in that paper).

Part 4: Proof of Theorem 5 with (4.7) and (4.8) replaced by

$$\begin{aligned} \frac{1}{C_0 |\log(r - r_0)|^{\frac{(k(r_0)-1)(p-1)}{2}}} &\leq T'(r) + \frac{r-r_0}{|r-r_0|} \leq \frac{C_0}{|\log(r - r_0)|^{\frac{(k(r_0)-1)(p-1)}{2}}}, \\ \frac{|r - r_0|}{C_0 |\log(r - r_0)|^{\frac{(k(r_0)-1)(p-1)}{2}}} &\leq T(r) - T(r_0) + |r - r_0| \leq \frac{C_0 |r - r_0|}{|\log(r - r_0)|^{\frac{(k(r_0)-1)(p-1)}{2}}}. \end{aligned}$$

The analogous statement in one space dimension with no perturbations is given in Theorems 1 and 2 in [22]. Thus, one needs to say how to adapt the analysis of the paper [22] to the present case. As in [25], three ingredients are needed in the proof:

- the trapping result stated in Theorem 2;
- the energy criterion stated in Corollary 4;
- the dynamics of equation (2.2) around a decoupled sum of solitons performed in [22] and presented in Part 3 above. Note that we have already adapted all these ingredients to the present context. With this fact, the adaptation given in [25] works here. See Part 4 page 371 in that paper for more details.

Part 5: Proof of (4.6), (4.7) and (4.8)

This part corresponds to the contributions brought in [7] in the one-dimensional case. The original statements in the one-dimensional case are given in Theorem 1.1 and Corollary 1.4 in that paper. Following Part 2 where we proved that (4.3) holds with (4.6) replaced by (4.10), a crucial step in one-space dimension was to prove that the solitons' centers satisfy the following ODE system for s large enough:

$$\frac{1}{c_1} \dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + O\left(\frac{1}{s^{1+\eta}}\right) \quad (4.11)$$

for some $\eta > 0$. In [7], we were able to use some ODE tools (particularly the Lyapunov convergence theorem) to further refine estimate (4.10) and prove that

$$\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 + o\left(\frac{1}{s^\eta}\right) \text{ as } s \rightarrow \infty.$$

Since we have for all $|d_1| < 1$ and $|d_2| < 1$

$$\|\kappa(d_1) - \kappa(d_2)\|_{\mathcal{H}} \leq C |\arg \tanh d_1 - \arg \tanh d_2|$$

(see estimate (174) page 101 in [20] for a proof of this fact), estimate (4.3) remains unchanged if one slightly modifies $\zeta_i(s)$ by setting $\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0$ which is the desired estimate in (4.6). That was the argument in one space dimension.

In our setting, since our perturbative terms contribute with additional exponentially decaying terms to the equation (see (3.5) and Part 2 above), we obtain that $\zeta_i(s)$ satisfy the same ODE system (4.11). Thus, the refinements of [7] hold here with no need for any further adaptations, and (4.6) holds.

As for estimates (4.7) and (4.8), let us point out that in one space dimension, they are derived in [7] as direct consequences of (4.6) on the one hand, and on the other hand a small improvement of the last argument of the paper [22] based on the equation in similarity variables. Since in our setting, (4.6) holds and the equation in similarity variables differs from the one dimensional case with exponentially decaying terms (see (3.5)), the same argument holds. See Section 2 in [7].

4.2 Existence and non-existence of characteristic points

Proceeding as in [7], we have the following result:

Theorem 7 (Existence of a solution with prescribed blow-up behavior at a characteristic point) *For any $r_0 > 0$ and $k \geq 2$, there exists a blow-up solution $u(r, t)$ to equation (1.6) with $r_0 \in \mathcal{S}$ such that*

$$\left\| \begin{pmatrix} w_{r_0}(s) \\ \partial_s w_{r_0}(s) \\ 0 \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^k (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0 \text{ as } s \rightarrow \infty, \quad (4.12)$$

with

$$d_i(s) = -\tanh \zeta_i(s), \quad \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 \quad (4.13)$$

for some $\zeta_0 \in \mathbb{R}$, where $\bar{\zeta}_i(s)$ is defined in (4.1).

Remark: When $N = 1$, we can take $r_0 = 0$. When $N \geq 2$, the multi-dimensional version $U(x, t) = u(|x|, t)$ has a sphere of characteristic points. Note also that this result uses the same argument as for Theorem 3, in particular, the analysis of the ODE system (4.11). If we simply want an argument for the existence of a blow-up solution with a characteristic point without caring about the number of solitons, then we have a more elementary proof which holds, however, only when g does not depend on $|x|$. See the remark following Theorem 9 below.

Remark: Note from (4.13) and (4.1) that the barycenter of $\zeta_i(s)$ is fixed, in the sense that

$$\frac{\zeta_1(s) + \cdots + \zeta_k(s)}{k} = \frac{\bar{\zeta}_1(s) + \cdots + \bar{\zeta}_k(s)}{k} + \zeta_0 = \zeta_0, \quad \forall s \geq -\log T(0). \quad (4.14)$$

Note that unlike in the one-dimensional case with a pure power nonlinearity treated in [7], we are unable to prescribe the barycenter. Indeed, our equation (1.6) is not invariant under the Lorentz transform.

Remark: We are unable to say whether this solution has other characteristic points or not. In particular, we have been unable to find a solution with \mathcal{S} exactly equal to $\{0\}$. Nevertheless, let us remark that from the finite speed of propagation, we can prescribe more characteristic points, as follows:

Corollary 8 (Prescribing more characteristic points) *Let $J = \{1, \dots, n_0\}$ or $J = \mathbb{N}$ and for all $n \in J$, $r_n > 0$, $T_n > 0$ and $k_n \geq 2$ such that*

$$r_n + T_n < r_{n+1} - T_{n+1}. \quad (4.15)$$

Then, there exists a blow-up solution $u(r, t)$ of equation (1.2) with $\{x_n \mid n \in J\} \subset \mathcal{S}$, $T(r_n) = T_n$ and for all $n \in I$,

$$\left\| \begin{pmatrix} w_{x_n}(s) \\ \partial_s w_{x_n}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k_n} (-1)^{i+1} \kappa(d_{i,n}(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0 \text{ as } s \rightarrow \infty,$$

with

$$\forall i = 1, \dots, k_n, \quad d_{i,n}(s) = -\tanh \zeta_{i,n}(s), \quad \zeta_{i,n}(s) = \bar{\zeta}_i(s) + \zeta_{0,n}$$

for some $\zeta_{0,n} \in \mathbb{R}$, where $\bar{\zeta}_i(s)$ is defined in (4.1).

Remark: Again, we are unable to construct a solution with $\mathcal{S} = \{r_n \mid n \in J\}$. When $N = 1$, we may take $r_0 \in \mathbb{R}$.

Proof of Theorem 7 and Corollary 8: First, note that thanks to condition (4.15) which asserts that the sections at $t = 0$ of the backward light cones with vertices (r_n, T_n) do not overlap, Corollary 8 follows from Theorem 7 by the finite speed of propagation. As for the proof of Theorem 7, we claim that it follows like in [7], since the ingredients of that paper are available here, thanks to the adaptations we performed in the previous sections:

- **the analysis of the ODE system** (4.11): let us emphasize the fact that we still encounter this system in our setting. Indeed, that system appears as a projection on the null modes of the linearization of equation (2.2) around the sum of decoupled solitons, and, as we said in Part 2 page 16, that equation differs from the pure power case, only with exponentially small terms (see (3.5)), which are absorbed in the $O(\frac{1}{s^{1+\eta}})$ in (4.11);

- **a reduction to a finite dimensional problem:** this is done thanks to the analysis of the dynamics of the equation in similarity variables (2.2) around the sum of decoupled solitons, which we did already for the proof of the isolatedness of characteristic points (see (i) of Theorem 5; see [22] for the analysis in one space dimension);

- **a topological argument to solve the finite dimensional problem:** this argument is based on a different formulation of Brouwer's Theorem. It is independent of the equation.

Note however that one argument of [7] does not work here: the argument that allows us to prescribe the barycenter of $\zeta_i(s)$. Indeed, that argument uses the invariance of the pure power wave equation under the Lorentz transform, which is no longer the case for equation (1.6). ■

Let us give in the following a criterion about the existence of characteristic points:

Theorem 9 (Existence and generic stability of characteristic points)

(i) **(Existence)** Let $0 < a_1 < a_2$ be two non-characteristic points such that

$$w_{a_i}(s) \rightarrow \theta(a_i)\kappa(d_{a_i}, \cdot) \text{ as } s \rightarrow \infty \text{ with } \theta(a_1)\theta(a_2) = -1$$

for some d_{a_i} in $(-1, 1)$, in the sense (3.2). Then, there exists a characteristic point $c \in (a_1, a_2)$.

(ii) **(Stability)** There exists $\epsilon_0 > 0$ such that if $\|(\tilde{U}_0, \tilde{U}_1) - (U_0, U_1)\|_{H^1_{\text{loc},u} \times L^2_{\text{loc},u}(\mathbb{R}^N)} \leq \epsilon_0$, then, $\tilde{u}(r, t)$ the solution of equation (1.6) with initial data $(\tilde{u}_0, \tilde{u}_1)(r) = (\tilde{U}_0, \tilde{U}_1)(x)$ if $r = |x|$ blows up and has a characteristic point $\tilde{c} \in [a_1, a_2]$.

Remark: This statement (valid for $N \geq 2$) is different from the original one (Theorem 2 in [24]) by two natural small facts: we take positive points a_1 and a_2 in (i), and we use the multi-dimensional norm in (ii) (of course, from the finite speed of propagation, it is enough to take a localized norm instead). When $N = 1$, we don't need the restriction $a_1 > 0$.

Remark: If one needs a quick argument for the existence of a blow-up solution for equation (1.6) with a characteristic point, then this theorem allows us to avoid the heavy machinery of [22], namely the linearization of equation (2.2) around the sum of decoupled solitons. Indeed, we have a more elementary argument, based on the knowledge of the blow-up behavior at a non-characteristic point on the one hand, and on (i) of this theorem on the one hand. However, such an argument uses the fact that solutions of the ODE (4.16) associated to (1.6) are also solution to (1.6) and this is possible only if g defined in (1.2) does not depend on $|x|$. For the statement with no perturbations, see Proposition 3 page 362 in [25]. For a further justification, see the Proof of Theorem 9 below.

Proof of Theorem 9: As in [25], there is no difficulty in adapting to the present context the proof of Theorem 2 of [24] given in Section 2 of that paper, except may be for some natural extensions to the radial case. Concerning the second remark following Theorem 9, the only delicate point is to find initial data (u_0, u_1) satisfying the hypothesis of (i) in Theorem 9. If g does not depend on $|x|$, then, any solution of the ODE

$$U'' = |U|^{p-1}U + f(U) + g(t, 0, U'), \quad (4.16)$$

is also a solution of the PDE (1.6), and it is enough to take initial data (u_0, u_1) with large plateaus of opposite signs. If g does depend on $|x|$, then this simple idea breaks down, and the existence of initial data with characteristic points holds thanks to Theorem 7. ■

We also have the following result which relates the existence of characteristic points to the sign-change of the solution:

Theorem 10 (Non-existence of characteristic points if the sign is constant) Consider $u(r, t)$ a blow-up solution of (1.6) such that $u(r, t) \geq 0$ for all $r \in (a_0, b_0)$ and $t_0 \leq t < T(r)$ for some real $0 \leq a_0 < b_0$ and $t_0 \geq 0$. Then, $(a_0, b_0) \subset \mathcal{R}$.

Remark: When $N = 1$, we don't need the restriction $a_0 \geq 0$.

Proof: This result follows from Theorem 3 above exactly as in one space dimension with no perturbations (i.e. for equation (1.6) with $(f, g) \equiv (0, 0)$). See the proof of Theorem 4 given in Section 4.2 in [24]. \blacksquare

A A local energy estimate for perturbations of the semilinear wave equation

Let us consider the following perturbation of equation (1.9):

$$\partial_t^2 U = \Delta U + |U|^{p-1}U + \lambda^{\frac{2p}{p-1}} f\left(\lambda^{\frac{-2}{p-1}} U\right) + \lambda^{\frac{2p}{p-1}} g\left(\lambda|x|, \lambda t, \lambda^{-\frac{p+1}{p-1}} \nabla U \cdot \frac{x}{|x|}, \lambda^{-\frac{p+1}{p-1}} \partial_t U\right), \quad (\text{A.1})$$

where $\lambda > 0$ and f and g satisfy (H_f) and (H_g) . The equation (A.1) is derived from equation (1.2) through the dilation

$$\lambda \mapsto U_\lambda(x, t) = \lambda^{\frac{2}{p-1}} U(\lambda x, \lambda t).$$

Using the technique of Shatah and Struwe [26] and introducing

$$\mathcal{E}(U(t)) = \int_{|x| < 1-t} \left[\frac{(\partial_t U(x, t))^2}{2} + \frac{(\nabla U(x, t))^2}{2} - \frac{|U(x, t)|^{p+1}}{p+1} - \lambda^{\frac{2p+2}{p-1}} F(\lambda^{\frac{-2}{p-1}} U(x, t)) \right] dx, \quad (\text{A.2})$$

where F is defined in (2.4). We obtain the following local energy estimate

Lemma A.1 (A local energy estimate for perturbations of equation (1.9))

For all $t \in [0, 1)$, we have

$$\mathcal{E}(U(t)) \leq C\mathcal{E}(U(0)) + C \int_0^t \int_{B_s} |U(\sigma, s)|^{p+1} d\sigma ds + C\lambda \int_0^t \int_{|x| < 1-s} |U(x, s)|^{p+1} dx ds + C\lambda^{\frac{2}{p-1}}.$$

where the lateral boundary is

$$B_{t_0} = \{(x, t) \mid 0 \leq t \leq t_0, |x| = 1 - t\}.$$

Proof: Classical calculation implies

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(U(t)) &= \underbrace{-\frac{1}{2} \int_{B_t} \left(\partial_t U - \frac{x}{|x|} \cdot \nabla U \right)^2 d\sigma}_{L_1(s)} - \underbrace{\frac{1}{2} \int_{B_t} \left(|\nabla U|^2 - \left(\frac{x}{|x|} \cdot \nabla U \right)^2 \right) d\sigma}_{L_2(s)} \\ &\quad + \int_{B_t} \left(\frac{|U|^{p+1}}{p+1} + \lambda^{\frac{2p+2}{p-1}} F(\lambda^{\frac{-2}{p-1}} U(x, t)) \right) d\sigma \\ &\quad - \lambda^{\frac{2p}{p-1}} \int_{|x| < 1-t} g\left(\lambda|x|, \lambda t, \lambda^{-\frac{p+1}{p-1}} \nabla U \cdot \frac{x}{|x|}, \lambda^{-\frac{p+1}{p-1}} \partial_t U\right) \partial_t U dx. \end{aligned}$$

Since $|\frac{x}{|x|} \cdot \nabla U| \leq |\nabla U|$, we can say that the term $L_2(s)$ is negative. By combining the estimate $|F(x)| \leq C(|x| + |x|^{p+1})$, the assumption (H_g) and the fact that the terms $L_1(s)$ and $L_2(s)$ are negative, we conclude that, for all $\lambda \in (0, 1]$

$$\frac{d}{dt}\mathcal{E}(U(t)) \leq C \int_{B_t} \left(\lambda^{\frac{2p}{p-1}} |U| + |U|^{p+1} \right) d\sigma + C\lambda \int_{|x|<1-t} \left(\lambda^{\frac{p+1}{p-1}} |\partial_t U| + |\nabla U|^2 + |\partial_t U|^2 \right) dx.$$

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(U(t)) &\leq C\lambda\mathcal{E}(U(t)) + C \int_{B_t} \left(\lambda^{\frac{2p}{p-1}} |U| + |U|^{p+1} \right) d\sigma \\ &\quad + C\lambda \int_{|x|<1-t} \left(\lambda^{\frac{p+1}{p-1}} |\partial_t U| + \lambda^{\frac{2p}{p-1}} |U| + |U|^{p+1} \right) dx \end{aligned}$$

So

$$\frac{d}{dt}\mathcal{E}(U(t)) \leq C\lambda\mathcal{E}(U(t)) + C \int_{B_t} |U|^{p+1} d\sigma + C\lambda \int_{|x|<1-t} |U|^{p+1} dx + C\lambda^{\frac{p+1}{p-1}}$$

Then

$$\begin{aligned} \mathcal{E}(U(t)) &\leq \mathcal{E}(U(0))e^{C\lambda t} + C \int_0^t e^{C\lambda(t-s)} \int_{B_s} |U|^{p+1} d\sigma ds \\ &\quad + C\lambda \int_0^t e^{C\lambda(t-s)} \int_{|x|<1-s} |U|^{p+1} dx ds + C\lambda^{\frac{p+1}{p-1}} \int_0^t e^{C\lambda(t-s)} ds \end{aligned}$$

so

$$\begin{aligned} \mathcal{E}(U(t)) &\leq \mathcal{E}(U(0))e^{C\lambda t} + C \int_0^t e^{C\lambda(t-s)} \int_{B_s} |U|^{p+1} d\sigma ds \\ &\quad + C\lambda \int_0^t e^{C\lambda(t-s)} \int_{|x|<1-s} |U|^{p+1} dx ds + C\lambda^{\frac{2}{p-1}} e^{C\lambda t} \end{aligned}$$

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